THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics

MATH2058 Honours Mathematical Analysis I
Tutorial 4

Date: 4 October, 2024

1. Find the limits of the following sequences defined by the recurrence relations:

   (a) $x_1 := \dfrac{3}{2}, x_{n+1} := 2 - \dfrac{1}{x_n}$;

   (b) $x_1 := 1, x_{n+1} := \sqrt{2x_n}$

2. (Exercise 3.4.12 of [BS11]) Show that if $\{x_n\}$ is unbounded, then there exists a subsequence $\{x_{n_k}\}$ such that $\lim \left(\dfrac{1}{x_{n_k}}\right) = 0$.

3. (Exercise 3.4.14 of [BS11]) Suppose $\{x_n\}$ is a sequence which is bounded from above. Let $s = \sup\{x_n\}$. Show that either $s = x_N$ for some $N \in \mathbb{N}$ sufficiently large, or that there is a subsequence x_{n_k} so that $x_{n_k} \to s$ as $k \to +\infty$.

4. (Exercise 3.4.15 of [BS11]) Let $\{I_n := [a_n, b_n]\}$ be a nested sequence of closed bounded intervals. For each $n \in \mathbb{N}$, let $x_n \in I_n$. Use the Bolzano-Weierstrass Theorem to prove the Nested Intervals Theorem.

Announcement: · HW2 posted on course website. Due 8/10 2359 on Gradescope.
· Quiz 1 returned. Total out of 30 pts.

1. Find the limits of the following sequences defined by the recurrence relations:

   (a) $x_1 := \frac{3}{2}, x_{n+1} := 2 - \frac{1}{x_n}$;

   (b) $x_1 := 1, x_{n+1} := \sqrt{2x_n}$

Pf: b): We will show $x_n \leq 2$ and $x_n \leq x_{n+1}$ for all $n \in \mathbb{N}$ by induction.

Base case: $x_1 = 1 < 2$, $x_2 = \sqrt{2} > 1 = x_1$ ✓

IH: Sps $x_k \leq 2$ and $x_k \leq x_{k+1}$ for some $k \in \mathbb{N}$.

Bded: $x_{k+1} = \sqrt{2x_k} \leq \sqrt{2 \cdot 2} = 2$.

Increasing: $x_{k+2} = \sqrt{2x_{k+1}} \geq \sqrt{2x_k} = x_{k+1}$ ✓

So since $\{x_n\}$ is bounded above and increasing, we conclude by Theorem 2.13 (Monotone Convergence Theorem) that x_n converges in $\mathbb{R}$ to some limit, say L.

Find L:

$$\lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} \sqrt{2x_n}$$

$$\Downarrow$$

$$L = \sqrt{2L} \Rightarrow L = \cancel{0}, 2$$

rejected b/c. $x_n \geq x_1 = 1 > 0$.

So $L = 2$. //

2. (Exercise 3.4.12 of [BS11]) Show that if $\{x_n\}$ is unbounded, then there exists a subsequence $\{x_{n_k}\}$ such that $\lim \left(\frac{1}{x_{n_k}}\right) = 0$.

Pf: Construct $\{x_{n_k}\}$ inductively. By unboundedness, for all $M \in \mathbb{R}$, we can find x_n s.t. $|x_n| > M$. So pick $n_1 \in \mathbb{N}$ s.t. $|x_{n_1}| > 1$.
Pick $n_2 \in \mathbb{N}$ s.t.

$$|x_{n_2}| > \max\{2, |x_1|, |x_2|, \ldots, |x_{n_1}|\}.$$

For each $k \in \mathbb{N}$, pick n_k s.t.

$$|x_{n_k}| > \max\{k, |x_1|, |x_2|, \ldots, |x_{n_{k-1}}|\}.$$

So we have $n_{k+1} > n_k$ and $\left|\frac{1}{x_{n_k}}\right| < \frac{1}{k} \to 0$ as $k \to \infty$. ✓

3. (Exercise 3.4.14 of [BS11]) Suppose $\{x_n\}$ is a sequence which is bounded from above. Let $s = \sup\{x_n\}$. Show that either $s = x_N$ for some $N \in \mathbb{N}$ sufficiently large, or that there is a subsequence x_{n_k} so that $x_{n_k} \to s$ as $k \to +\infty$.

Pf: Sps it's not the case that $x_N = s$ for some $N \in \mathbb{N}$. By ε-characterization of supremum, taking $\varepsilon = 1$, $\exists\, n_1 \in \mathbb{N}$ s.t. $x_{n_1} > s-1$.

IH: Sps. $\exists\, x_{n_1}, \ldots, x_{n_k}$ s.t. $x_{n_l} > s - \frac{1}{l}$ for $l = 1, \ldots, k$.

If we can show $s = \sup\{x_n : n > n_k\}$, then we would be done b/c. we could find n_{k+1} s.t. $x_{n_{k+1}} > s - \frac{1}{k+1}$ and $n_{k+1} > n_k$

Notice that $\{x_n : n > n_k\} \subseteq \{x_n\}$, so $v = \sup\{x_n : n > n_k\} \leq s$

Now suppose $v < s$. Since by assumption $x_n < s$, restricting n to range from $n = 1, \ldots, n_k$, we have

$$\max\{v, \max\{x_n : n = 1, \ldots, n_k\}\} < s$$

So we have found an u.b. of $\{x_n\}$ strictly less than s, hence we have a contradiction

So $v = s$. //.

4. (Exercise 3.4.15 of [BS11]) Let $\{I_n := [a_n, b_n]\}$ be a nested sequence of closed bounded intervals. For each $n \in \mathbb{N}$, let $x_n \in I_n$. Use the Bolzano-Weierstrass Theorem to prove the Nested Intervals Theorem.

Pf: Easy to show $\{x_n\}$ is bounded since $a_1 \leq x_n \leq b_1$ for all $n \in \mathbb{N}$. So by Bolzano-Weierstrass, there exists a subsequence $\{x_{n_k}\}$ that converges in $\mathbb{R}$ to some limit, say L.

Remains to show $L \in I_k$ for each $k \in \mathbb{N}$. Let $k \in \mathbb{N}$ be fixed. Then for each $l \geq k$, we can find $n_l \geq n_k \geq k$ s.t.

$$a_k \leq x_{n_l} \leq b_k$$

$$\Big\downarrow \; l \to \infty$$

$a_k \leq L \leq b_k$, so $L \in I_k$ for each $k \in \mathbb{N}$.

Uniqueness when $\lim_{n\to\infty} (b_n - a_n) = 0$ is the same argument. //